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# DISCRETE WAREHOUSE PROBLEM

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## Discrete Warehouse Problem †

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### Abstract

A *discrete warehouse* is a collection of objects (robot and obstacles) which are allowed to move along the grid edges of a two-dimensional grid. In this paper, we consider motion planning problems of a unit-square robot moving on a square grid among unit-square obstacles, which are movable. In such a setup one is allowed to move some of the obstacles in order to navigate the robot between an initial and a final position of the robot in the warehouse. The final positions of the obstacles are unimportant for our problems. We consider two forms of obstacle manipulations: (a) *remote*, when the obstacles are moved by a remote mechanism, and (b) *contact*, when the obstacles are moved only by direct contact of the robot. We present necessary and sufficient conditions for the existence of a motion in both cases, and propose efficient algorithms for constructing feasible motions. Thus, we establish the first known class of tractable motion planning problems with multiple movable obstacles.

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## 1. Introduction

Computational geometry, or more generally, algorithmics, is a newcomer to the multifaceted area of robotics. The repertoire of robotics problems presently treated by computational geometry is admittedly narrow in view of the total scope of robotics. Most of the recent efforts have been towards studying the problems of motion planning. In this way there are basically two different options: (a) motion planning with fixed obstacles (or, obstacle-avoiding motion planning) and, (b) motion planning with movable obstacles (or, obstacle-manipulation motion planning). First of which has been called the Piano mover's problem by Schwartz and Sharir in an influential series of papers [3-6] and has been studied by many authors since (see [9] for an overview of some of these results). In its general form the Piano Mover's problem is defined as follows: Given a fixed configuration of obstacles in the the Euclidean space and an initial and a final configuration of a robot, determine whether there exists a continuous motion of the robot between its initial and final configuration during which it does not intersect any of the fixed obstacles. This problem is known to be solvable in polynomial time and space.

In contrast, the *coordinated motion planning* problem defined by Hopcroft et. al. [1], relates to motion planning with obstacle manipulation. The problem is to determine if there exists a continuous motion of a collection of disjoint movable objects constrained to move within an enclosed region, between the initial and final configurations of the objects during which they do not penetrate either the "walls" of the enclosing box, or each other. They show that even the restricted two-dimensional version of this problem for arbitrarily many rectangles in a rectangular region is PSPACE-hard. More recently, a restricted version of the problem, involving translations of one movable polygonal obstacle, was shown to be tractable [8].

In this paper, we formulate a new class of motion planning problems similar to the "warehouseman's problem" [1]. We consider the following environment (see Figure 1): A warehouse containing a robot  $B$  and a collection of movable and fixed obstacles  $O$ . For our purposes we assume that obstacles and the robot are all square shaped with unit dimensions, and are placed on the unit square grid. In an abstract formulation we view the warehouse as a subset (subgraph) of the unit square grid

(graph). Each grid point is either empty, contains a movable object, contains a fixed object, or contains the robot. Motion of robot and the obstacles is restricted to grid edges of the warehouse.

We define two cases for such a system depending on the manner in which obstacles are moved: (a) by a *contact* motion, or (b) by a *remote* motion (e.g., a mechanical arm). For a contact movement the obstacle being moved has to be adjacent to the robot. In this case, robot  $B$  is a special object which can move on its own and all other obstacles of  $O$  can move only by its consequence. For a remote movement the obstacles are moved by a remote mechanism, called the *grip*, and need not be adjacent to the robot. In this case, robot  $B$  is like just another obstacle of  $O$ . In both cases movement of an obstacle is valid if it moves in an adjacent location on the grid which is empty. This configuration of obstacles and robot resembles a warehouse where items are stored in movable shelves and have to be retrieved by a robot moving on the grid edges. We permit the robot to access an item from a shelf by being at one of the four empty grid points around it.

For both the contact and remote motions, we study two kinds of problems: (1) how to move the robot from an initial position to a final position in the warehouse; (2) how to construct a "clearing" (i.e., a free path) between two locations of the warehouse (one of these points may be a "door" on the boundary). For both problems we present necessary and sufficient conditions for the existence of a solution and propose efficient algorithms for constructing feasible motions.

In Section 2, we give the basic notation and some of the definitions required for describing our results. In Section 3, we discuss the case of remote motions. Finally, in Section 4, we outline our approach for contact motions.

## 2. Problem Definition

We denote the warehouse  $W$  by a rectilinear region of the unit square grid of size  $m \times n$ . At any instant, grid point  $(i, j)$  of the warehouse  $W$  is either *empty*, *movable*, *fixed*, or has a *robot* ( $e, m, f$ , or  $r$ , respectively). We represent the *type* of  $(i, j)$  by  $T(i, j) = x$ , where  $x \in \{e, m, f, r\}$ . For our considerations only the relative positions of obstacles is important, since all the obstacle items are indistinguishable. Configuration  $R$  of the warehouse is specified by an  $m \times n$  type matrix  $T$  which is like a

snapshot of  $W$  describing the types of items for all  $m \times n$  grid points of  $W$ .

Robot  $B$  is located at the grid point of type  $r$  and its coordinates are  $(b_x, b_y)$ . We call,  $S$ , a set of obstacle items *H-consecutive* if they are consecutive on a row of the array  $W$  and are all of type  $m$ . Similarly, items of  $S$  are *V-consecutive* if they are consecutive on a column of array  $W$  and all are of type  $m$ .  $|S|$  denotes the number of items in  $S$ . Let  $S_L(i, j)$  (resp.  $S_R(i, j)$ ) represent the maximal H-consecutive set of items to the left (resp. right) of grid position  $(i, j)$  and let  $S_U(i, j)$  (resp.  $S_D(i, j)$ ) represent maximal V-consecutive set of items in the upward (resp. downward) direction of grid position  $(i, j)$ . In the following we describe five elementary motions (EM) permitted (in terms of the above notation) for remote and robot motion (see Figure 2).

*Elementary Contact Manipulation:*

- (1) *Free* motion: the robot can move to an adjacent  $e$ -type grid point along the grid edges of  $W$ .
- (2) *Push* motion: the robot can push sets  $S_L(b_x, b_y)$  and  $S_R(b_x, b_y)$  (resp.  $S_U(b_x, b_y)$  and  $S_D(b_x, b_y)$ ) of  $k$  or less *H-consecutive* (resp. *V-consecutive*) items adjacent to it in a row (resp. column) by one position provided there is an empty space on the row (resp. column) within  $k$  spaces of it in the direction of push. Then, the robot moves one unit in the direction of push.
- (3) *Pull* motion: the robot can pull sets  $S_L(b_x, b_y)$  and  $S_R(b_x, b_y)$  (resp.  $S_U(b_x, b_y)$  and  $S_D(b_x, b_y)$ ) of  $k$  or less *H-consecutive* (resp. *V-consecutive*) items adjacent to it in a row (resp. column) by one position provided there is an empty space right next to it on that row (resp. column) in the direction of pull.
- (4) *Slide* motion: the robot can slide sets  $S_L(b_x \pm 1, b_y)$  and  $S_R(b_x \pm 1, b_y)$  (resp.  $S_U(b_x, b_y \pm 1)$  and  $S_D(b_x, b_y \pm 1)$ ) of  $k$  or less *H-consecutive* (resp. *V-consecutive*) items along the row (resp. column) with respect to position adjacent to it on the column (resp. row) provided there is an empty space within  $k$  spaces of  $(b_x \pm 1, b_y)$  (resp.  $(b_x, b_y \pm 1)$ ).

*Elementary Remote Manipulation:*

- (5) *Remote motion:* Any set of  $k$  or less  $H$ -consecutive (resp.  $V$ -consecutive) items can be moved along a row (resp. column) provided there is an empty space within  $k$  spaces of its either end points.

Consider an  $m \times n$  grid warehouse  $W$ , with  $1 \leq x \leq m$  and  $1 \leq y \leq n$ . Grid point  $(x, y)$  is said to be *adjacent* to grid points  $(x-1, y)$ ,  $(x+1, y)$ ,  $(x, y-1)$ , and  $(x, y+1)$  (if they exist in  $W$ ). Let  $R_0$  denote the initial configuration of  $W$  which is specified by the type-matrix  $T_0$ .

A configuration  $R_j$  is said to be *reachable* from a configuration  $R_i$ , if it can be obtained from  $R_i$  by sequence of EMs (i.e., elementary motions). For simplicity, we say a configuration  $R$  is reachable, to mean  $R$  is reachable from the initial configuration  $R_0$ . The *robot displacement problem (RDP)* is the problem of obtaining a reachable configuration with the robot being at a specified grid point  $P_f$ . In a configuration  $R$  there is a *clearing* from point  $P_a$  to point  $P_b$ , if there exists a sequence of distinct points  $P_a = P_{\pi_1}, P_{\pi_2}, \dots, P_{\pi_k} = P_b$  such that  $P_{\pi_i}$  is adjacent to  $P_{\pi_{i+1}}$ , for  $i < k$ , and  $T(P_{\pi_j}) = e$ , for all  $j$ . The *clearance construction problem (CCP)* involves obtaining a reachable configuration  $R$ , which has a clearing between the two specified points  $P_0$  and  $P_f$ .

Grid points of type  $e$  and  $m$  are called *free grid points* and grid points of type  $f$  are called *rigid grid points*. We partition free grid points of  $W$  into equivalence classes  $C_0, \dots, C_t$ . If a free grid point is placed in class  $C_i$  then all the adjacent free grid points are also placed in  $C_i$  (see Figure 3). Rigid points form boundaries of equivalence classes. Note that type of a rigid point remains unchanged under an elementary motion EM, thus:

**Fact 1 :** Equivalence classes are preserved under EM.

Consider two grid points  $P_a$  and  $P_b$ . A shortest path  $S_{ab}$  between  $P_a$  and  $P_b$  is a minimum cardinality sequence  $P_a = P_{\pi_0}, P_{\pi_1}, \dots, P_{\pi_k} = P_b$  such that  $P_{\pi_i}$  is adjacent to  $P_{\pi_{i+1}}$ , for  $0 \leq i \leq k-1$ , and  $P_{\pi_j}$  is free, for  $1 \leq j \leq k-1$ . If such a sequence does not exist (i.e.,  $P_a$  and  $P_b$  belong to different classes) then we write  $|S_{ij}| = \infty^*$ . The number of  $e$ -type grid points (holes) in class  $C_i$  is denoted by  $h_i$ . By virtue of

Fact 1,  $h_i$ 's remain the same under EM, i.e., holes are neither destroyed nor transferred.

In the next two sections, we study RDP and CCP for the cases of remote motions and contacts motions.

### 3. Remote Motion

In this section we study remote motion planning. The *grip* (or, mechanical arm) is allowed to move a movable shelf into an adjacent empty spot, or equivalently, to "move" an empty spot into an adjacent movable shelf. We will adopt the later convention, for it simplifies the proofs. (Using metaphor, the e/m type movement is similar to electron/hole movement in semiconductors.) The (XY-)grip has only horizontal and vertical movements, resembling the pen movement in an XY-plotter. In compliance with the grid environment  $W$  and the XY-grip, all distances are measured in  $L_1$  metric. The following operation, performed by the grip, changes a configuration:

**OP:** if  $T(P_1) = e$  and  $T(P_2) = m$  and  $P_1$  and  $P_2$  are adjacent then  $T(P_1) := m$  and  $T(P_2) := e$ .

First we study robot displacement problem (RDP). Consider the initial configuration  $R_0$  with a distinguished movable obstacle  $O$  at the grid point  $P_0$ . We aim to obtain a reachable configuration with  $O$  at a specified grid point  $P_f$ . Thus, an instance of RDP is specified by a pair  $(P_0, P_f)$  of grid points (and an initial configuration  $R_0$ ). To simplify the case analysis below, we assume  $P_0 \neq P_f$ .

**Lemma 1:** An instance  $(P_0, P_f)$  of RDP has a solution if and only if  $P_0$  and  $P_f$  belong to  $C_0$  and  $h_0 \geq 1$ , where  $C_0$  is the class to which  $P_0$  belongs.

**Proof:** (*only if*) If  $h_0 = 0$  then **OP** is not possible. Therefore configuration of  $C_0$  remains unchanged.

(*if*) Consider a path  $(P_h = P_{\pi_1}, \dots, P_{\pi_a} = P_0)$  from an e-type grid point (a hole)  $P_h$  to  $P_0$ . By a sequence of **OP**'s (interchanging type of  $P_{\pi_i}$  and  $P_{\pi_{i+1}}$ , for  $0 \leq i \leq a-2$ ) the hole is brought adjacent to  $P_0$ . Now, consider a path  $(P_0 = P_{\psi_1}, \dots, P_{\psi_b} = P_f)$  from  $P_0$  to  $P_f$ . If  $P_{\psi_i}$  contains  $O$  (the distinguished obstacle) and one of the grid points adjacent to it has a hole then we can move  $O$  to  $P_{\psi_{i+1}}$ , as shown in Figure 4. Since  $P_{\pi_{a-1}}$  is empty and it is adjacent to  $P_0$  then (inductively) we can move  $O$  to  $P_f$ .  $\square$

\* In implementation, choose  $\infty = mn + 1$ .



As outlined in Lemma 1, an algorithm for RDP involves obtaining a shortest path  $S_{h_0}$  between  $P_h$  and  $P_0$  and a shortest path  $S_{0f}$  between  $P_0$  and  $P_f$ . From the previous discussion and the algorithm in [2], we conclude:

**Theorem 1 :** An instance of RDP can be solved in  $O(|S_{h_0}| |S_{0f}| \log(|S_{h_0}| |S_{0f}|))$ .

Next we consider clearance construction problem (CCP). An instance of the problem involves two points  $P_0$  and  $P_f$  (and an initial configuration  $R_0$ ).

**Lemma 2 :** There exists a reachable configuration with a clearing from  $P_0$  to  $P_f$  if and only if  $h_0 \geq |S_{0f}|$ , where  $C_0$  is the class to which  $P_0$  belongs.

**Proof :** (only if) Assume  $h_0 < |S_{0f}|$ .

Case 1)

$|S_{0f}| = \infty$ : In this case,  $P_0$  and  $P_f$  belong to different classes. Boundary of  $C_0$  consists of a set of rigid points, and thus, cannot be changed under OP. Therefore, in any reachable configuration  $|S_{0f}| = \infty$  (see Fact 1).

Case 2)

$|S_{0f}| \neq \infty$ : In this case,  $P_0$  and  $P_f$  belong to the same class  $C_0$ . In a clearing from  $P_0$  to  $P_f$  there must be a path of  $e$ -type grid points from  $P_0$  to  $P_f$ . Thus the number  $h_0$  of  $e$ -type grid points in  $C_0$  must be at least equal to the length of the path. Since  $|S_{0f}|$  is the length of the shortest path then there is no solution.

(if) Assume  $h_0 \geq |S_{0f}|$ . We will show how to obtain a reachable configuration  $R_f$  with a clearing from  $P_0$  to  $P_f$ . Consider class  $C_0$  containing  $P_0$  and  $P_f$ . Let  $P_0 = P_{\pi_0}, P_{\pi_1}, \dots, P_{\pi_k} = P_f$  be a shortest path from  $P_0$  to  $P_f$ , such that  $P_{\pi_i}$  is adjacent to  $P_{\pi_{i+1}}$ , for  $0 \leq i \leq k-1$ . Inductively, assume there exist a reachable configuration  $R_i$  with  $T(P_{\pi_0}), \dots, T(P_{\pi_i}) = e$ ; this is certainly the case for  $i=0$ . Next, we show that there exist a configuration  $R_{i+1}$  reachable from  $R_i$  with  $T(P_{\pi_0}), \dots, T(P_{\pi_{i+1}}) = e$ . Consider an  $e$ -type point  $P_j$  ( $\neq P_{\pi_0}, \dots, P_{\pi_{i+1}}$ ) in  $C_0$ . Such a point must exist since  $h_0 \geq |S_{0f}|$  and  $i \leq k$ . By virtue of

Lemma 1, there is a configuration with  $T(P_l) = e$ , where  $P_l$  is a grid point adjacent to  $P_{\pi_i}$  with  $\pi_i \in \{\pi_0, \dots, \pi_i\}$  (essentially by "switching" type of  $P_j$  and type of  $P_l$ ). By sequence of  $\pi_i - \pi_l$  OP's we obtain a configuration with  $T(P_{\pi_{i-1}}), \dots, T(P_{\pi_{i-1}}) = e$ . One more OP results in configuration  $R_{i+1}$  with  $T(P_{\pi_{i-1}}) = e$  (by "switching" type of  $P_{\pi_i}$  and type of  $P_l$ , see Figure 5). When  $k = i-1$ , there is a clearing from  $P_0$  to  $P_f$ .  $\square$

Lemma 2 (if part) dictates an effective procedure for constructing a clearing. Next, we will propose an efficient implementation of the outlined procedure. When the grip is in motion it is either *loaded* (moving a movable object) or *unloaded* (moving toward an object to be moved). Let  $\tau$  be the total time for constructing a clearing. We can write  $\tau = \tau_l + \tau_u$ , where  $\tau_l$  and  $\tau_u$  are the total time the grip is loaded and unloaded, respectively. When the grip is loaded it moves much slower than when it is unloaded, that is,  $\tau_l \gg \tau_u$ . Thus, the primary objective is to minimize  $\tau_l$  and the secondary objective is to minimize  $\tau_u$ . Next, we present an optimal algorithm for minimizing  $\tau_l$  and a 2-approximation algorithm for minimizing  $\tau_u$  assuming a shortest path  $S_*$  from  $P_0$  to  $P_f$  has been obtained.

Let  $\lambda_l$  be the time that it takes to move a movable shelf into an adjacent empty spot. We can write  $\tau_l = \lambda_l \eta$ , where  $\eta$  is the total number of OP's. We aim to minimize  $\eta$  (denoted by  $\eta_*$ ), or equivalently,  $\tau_l$ .

We assume a shortest path  $S_*$  from  $P_0$  to  $P_f$  has been obtained. With reference to  $R_0$ , we construct a (complete) weighted bipartite graph  $G = (V_e, V_p, E)$ , where  $V_e$  is the set of  $e$ -type grid points in  $C_0$  and  $V_p$  correspond to vertices of  $S_*$ . Weight  $w(e)$  of an edge  $e = (v_1, v_2)$  is the length of a shortest path (as before, in  $L_1$  metric) from  $v_1$  to  $v_2$ . Consider a matching  $M = \{e_1, \dots, e_{|S_*|}\}$  in  $G$ . A motion  $\mu_M$  corresponding to  $M$  is, for each  $e_i = (v_{i_1}, v_{i_2}) \in M$ , a sequence of  $w(e_i)$  OP's resulting in a configuration with  $T(v_{i_1}) = e$ . The number of OP's performed in  $\mu_M$  is denoted by  $\eta_M$ .

**Lemma 3:** For a minimum weighted matching  $M$  in graph  $G$ ,  $\eta_M = \eta_*$ .

**Proof:** Consider the path  $S_* = P_{\pi_0}, \dots, P_{\pi_k}$ . In the final configuration  $T(P_{\pi_i}) = e$ , for all  $i$ . Let  $P_{\sigma_i}$  be the  $e$ -type grid point whose type was switched, by a sequence of OP's, with the type of  $P_{\pi_i}$ . Clearly,

$\eta_* = \sum_{i=0}^{i=k} |S_{\pi_i \sigma_i}|$ , where  $S_{\pi_i \sigma_i}$  is a shortest path from  $P_{\pi_i}$  to  $P_{\sigma_i}$ . Since  $M$  is a minimum weighted matching in  $G$ , and for each pair  $(P_{\pi_i}, P_{\sigma_i})$  there is an edge with weight  $|S_{\pi_i \sigma_i}|$  in  $G$  then  $w(M) \leq \sum_{i=0}^{i=k} |S_{\pi_i \sigma_i}|$ , or equivalently,  $\eta_M \leq \eta_*$ .  $\square$

The motion  $\mu_M$  corresponding to  $M$  can be obtained by constructing  $G$  and obtaining a minimum-weighted matching in  $G$ . Using the geometric nature of the matching problem and employing the matching algorithm of [7], we obtain the following theorem.

**Theorem 2:** The motion  $\mu_M$  can be calculated in  $O(|V_e|^{2.5} \log(|V_e|)^2)$  time.

Given a matching  $M$ , we aim to minimize the extra movement, that is,  $\tau_u = \lambda_u \beta$  where  $\lambda_u$  is the time it takes to move the grip from one grid point to an adjacent grid point and  $\beta$  is the total number of grid points the grip traverses. Let  $\beta^*$  denote the optimal  $\beta$ . Consider an initial position  $P_s$  of the grip (see Figure 5). When the matching is fixed (solid edges) the grip must move from  $P_s$  to one of the holes at  $P_h$ . The hole is brought to one of the position in the path. This process is repeated until all the holes are brought to the path.

Let  $(P_1, P_2), \dots, (P_{m-1}, P_m)$  be the matching constructed in the previous step. Consider the greedy unloaded motion  $\mu_F: (P_s = P_0, P_1), \dots, (P_{m-2}, P_{m-1})$ . Let  $l_i$  denote the length (in  $L_1$ -metric) between  $P_i$  and  $P_{i+1}$ . The number of unloaded movements corresponding to  $\mu_F$  is  $\beta_F = l_0 + l_2 + \dots + l_{m-2}$ .

**Lemma 4:**  $\beta_F \leq 2\beta^*$ .

**Proof:** We assume the path does not contain an empty point, i.e.,  $l_1, l_3, \dots, \neq 0$  (other case is handled similarly).  $\beta^*$  is movement from  $P_0$  to some hole  $P_i$  (odd  $i$ ) plus some matching between the holes (odd  $P_i$ ) and points on the path (even  $P_i$ ).

Thus:

$$\beta^* - l_0^* \geq l_1 + l_3 + \dots + l_{m-1} = |M| \quad (1)$$

where  $l_0^*$  is the shortest path from  $P_0$  to some hole  $P_i$ . Also,  $l_{2k+1} + 1$ , due to the triangle inequality.

Thus,

$$\beta_F - l_0 (= l_2 + l_4 + \dots + l_{m-2}) \leq |M| - l_1 + \frac{m}{2}$$

$$\beta_F - l_0 + l_1 - \frac{m}{2} \leq |M| \quad (2)$$

From Equations (1) and (2):

$$\beta^* - l_0^* + l_0 + \frac{m}{2} - l_1 \geq \beta_F$$

Since  $\beta^* \geq m/2$  and  $l_0^* \leq l_0$ , then  $2\beta^* \geq \beta_F$ .  $\square$

Thus, the greedy motion  $\mu_F$  is provably good and is readily obtained.

**Theorem 3:** The motion  $\mu_F$  can be calculated in  $O(m)$  time.

The above theorem illustrates that the discrete warehouse problem under remote motion has a polynomial algorithmic solution, which establishes it as the first class of tractable problems with multiple moveable obstacles. In the next section, we show that the case of discrete warehouse problem with contact motions is also tractable.

#### 4. Contact Motion

In this section, we consider motion planning in the presence of contact obstacle manipulations. As discussed before, the robot  $B$  must be adjacent to a movable obstacle in order to move it. Moreover, there must be an empty location in line with the intended direction of motion. It is easy to observe that movements of the robot in such an environment is more restrictive than in the remote environment. Here, we consider two different kinds of robots; one which can push, pull and slide (called an  $F$ -robot) and the other which can only push the adjacent obstacles (called an  $H$ -robot). We discuss necessary and sufficient conditions for RDP and CCP for these both cases.

Unlike the case of remote motions not every empty location of  $W$  is reachable by the robot. More specifically, a robot can only access empty locations which lie in a "circle" of radius  $k$  around it. The definition of the "circle" region is different for both  $F$ - and  $H$ -robots. In the case of an  $H$ -robot (one with only pushing capability) the "circle" contains locations no more than  $k$  units apart from  $(b_x, b_y)$  on the row and column containing  $B$ . In the case of an  $F$ -robot the "circle" contains locations no more than  $k$  units away from  $(b_x, b_y)$  on the rows (resp. columns) containing  $B$ , above (resp. left)  $B$ , and below (resp.

right)  $B$  (see Figure 6). If there is no empty location in the influence circle of the robot then it is stuck in that configuration. We redefine  $h_i$  as the maximum number of holes that lie in the circle of influence over all the configurations. It is easy to see that a robot does not endanger its chances of movement at any time by moving all empty locations in its circle next to itself. This can be achieved by using the push and slide EMs. It is also easy to see that the pull motions send the empty locations away from the "center" and so are not useful in this respect.

**Fact 2 :** Pull motions do not help in using the empty holes for motion.

First we consider *RDP*. Consider the initial configuration  $R_0$  with  $B$  at  $P_0$ . We obtain a reachable configuration with  $B$  at  $p_f$ . Due to the freedom of manipulation, an  $F$ -robot is similar to a remote robot and it can manage to move about with the help of only one hole, whereas an  $H$ -robot needs a new hole every time it advances a step. Then we have the following lemmas.

**Lemma 6 :** An instance  $(P_0, P_f)$  of *RDP* for an  $F$ -robot has a solution if and only if  $P_0$  and  $P_f$  belong to  $C_0$  and  $h_0 \geq 1$ , where  $C_0$  is the class to which  $P_0$  belongs.

**Lemma 5 :** An instance  $(P_0, P_f)$  of *RDP* for an  $H$ -robot has a solution if and only if  $P_0$  and  $P_f$  belong to  $C_0$  and  $h_0 \geq |S_{of}|$ , where  $C_0$  is the class to which  $P_0$  belongs.

The existence of holes in the circle of robot can be checked in time  $O(k)$ . Then, using a procedure similar to that for the remote motion, we have the following theorem.

**Theorem 4 :** An instance of *RDP* for contact motion can be solved in  $O(|S_{h0}| |S_{of}| \log(|S_{h0}| |S_{of}|))$ .

Next, we consider the clearance construction problem (see Figure 7). In summary, We have the following theorems:

**Theorem 5 :** There exists a clearing from  $P_0$  to  $P_f$  in an instance of *CCP* for a contact motion if and only if there are at least  $|S_{of}|$  holes in the circles of influence on the path.

Again by using the Euclidean matching algorithm of [7] we can construct the minimum weight matching of the holes in the circles of influence to the points of the path. Thus,

**Theorem 6 :** The solution to *CCP*, when it exists can be found in  $O(|S_{of}|^{2.5} \log^2 |S_{of}|)$ .

## 5. Discussion and Open Problems

In this paper we have introduced a new class of motion planning problems, called the discrete warehouse problem, for robots on a two dimensional grid in the presence of movable obstacles. Our results demonstrate that problems of this class are tractable, unlike the "warehouseman's problem" of [1].

Besides the issues about the existence of a motion of a robot and the feasibility of a clearing, as discussed in this paper, one can address other aspects of this problem. For example, in some situations it is important to know the shortest time clearing, where the goal is to reduce the total time required for a clearing and not necessarily the shortest clearing. Also, questions about motions with minimum number of elementary motions is also a natural question, which we suspect to be *NP*-Complete. Another interesting direction for research would be to consider this problem in a dynamic framework, where the states of the shelves of the warehouse changes dynamically by external interactions. The problem of finding lower bounds for the various problems is another open question.

This paper, establishes a framework for considering motion planning in the presence of movable obstacles which form a significant part of the robotics problems occurring in real life.

## References

- [1] Hopcroft, J., J. Schwartz, and M. Sharir, "On the Complexity of Motion Planning for Multiple Independent Objects: PSPACE Hardness of the Warehouseman's Problem," *Int. J. Robotics Res.*, vol. 3, pp. 72-89, 1984.
- [2] Lee, D. T. and C. K. Wong, "Voronoi Diagrams in  $L_1$  ( $L_\infty$ ) metrics with 2-Dimensional Storage Applications," *SIAM J. of Comp.*, vol. 9, no. 1, pp. 200-211, 1980.
- [3] Schwartz, J. T. and M. Sharir, "On the piano movers' problem: II. General techniques for computing topological properties of real Algebraic manifolds," *Adv. Appl. Math.*, vol. 4, pp. 298-351, 1983.

- [4] Schwartz, J. T. and M. Sharir, "On the piano movers' problem: III. Coordinating the motion of several independent bodies: The special case of circular bodies amidst polygonal barriers," *Robotics Research*, vol. 2, pp. 46-75, 1983 .
- [5] Schwartz, J. T. and M. Sharir, "On the piano movers' problem: I. The special case of a rigid polygonal moving amidst polygonal barriers," *Comm. Pure Appl. Math.*, vol. XXXVI, pp. 345-398, 1983.
- [6] Sharir, M. and E. A. Sheffi, "On the piano movers' problem: IV. Various Decomposable Two-Dimensional Motion Planning Problems," *Comm. on Pure and Appl. Math*, vol. XXXVII, pp. 479-493, 1984.
- [7] Vaidya, P., "Geometry Helps in Matching," *Proc. of 20th STOC*, pp. 422-425, 1988.
- [8] Wilfong, G., "Motion Planning in the Presence of Movable Obstacles," *To appear in the Proc. of 4th Sym. on Comp. Geo.*, 1988.
- [9] Yap, C. K., "Algorithmic motion planning," in *Advances in Robotics, Volume 1: algorithmic and geometric issues*, J.T. Schwartz and C.K. Yap, Lawrence Erlbaum Associates, 1985.

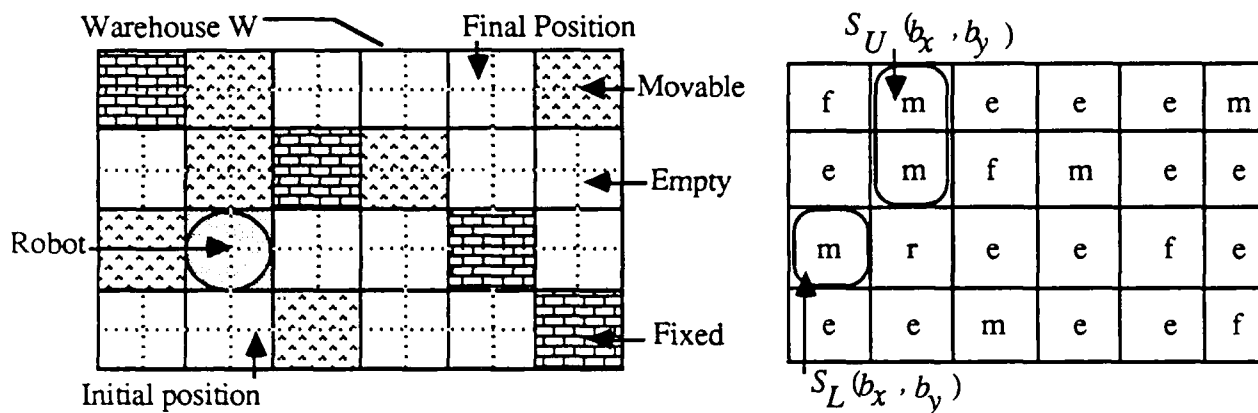


Fig. 1: Discrete Warehouse Problem

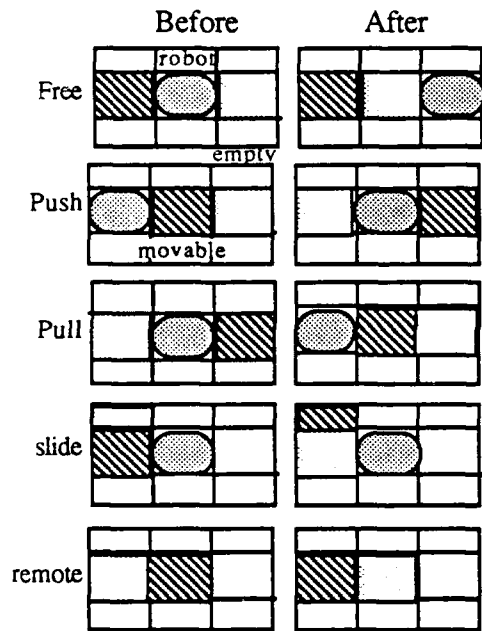


Fig. 2: Various Elementary Motions

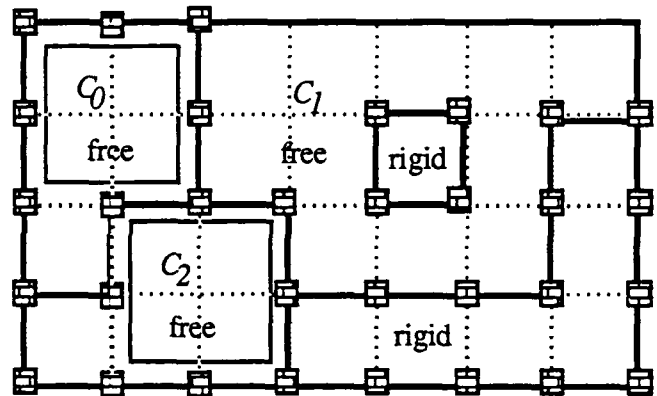


Fig. 3: Equivalence classes

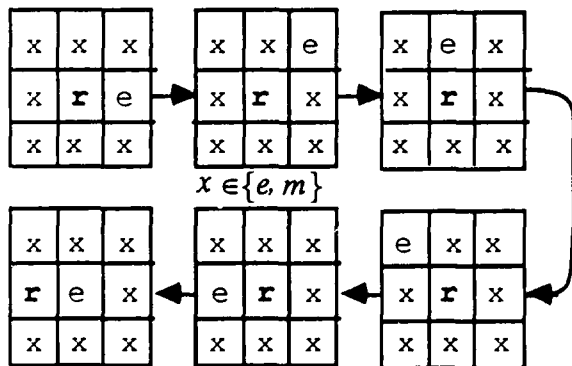


Fig. 4: Advancing the distinguished obstacle r

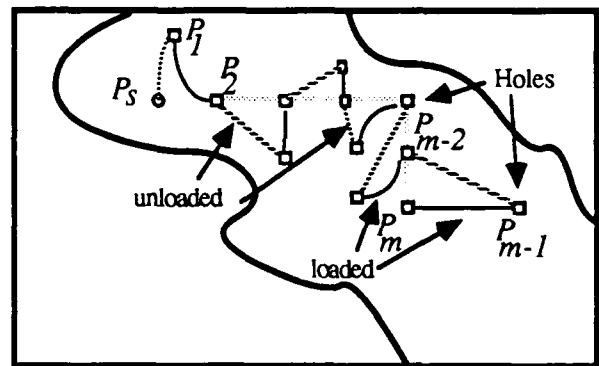


Fig. 5: Bringing holes to a path



